POLYNOMIAL SOLUTIONS OF BINARY QUADRATIC DIOPHANTINE EQUATION

$x^2 - a(t)y^2 - 2b(t)x + 2ya(t)c(t) = 0$

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ABSTRACT:

Let a (t), b (t), c (t) be polynomials in Z[x]. In this paper we consider the number of polynomial solutions of Diophantine equation $x^2 - a(t)y^2 - 2b(t)x + 2ya(t)c(t) = 0$ .We also obtain recurrence relations and some properties on the polynomial solutions.

Keywords: Polynomial solutions, Pellian equation, Diophantine equation

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INTRODUCTION

Binary quadratic diophantine equations offer an unlimited field for research because of their variety [1-4]. In the context one may refer [5-26] .This communication concerns with yet another interesting binary quadratic equation $x^2 - a(t)y^2 - 2b(t)x + 2ya(t)c(t) = 0$ for determining it's infinitely many non zero polynomial solutions. Also a few interesting relations among the solutions are presented.

METHOD OF ANALYSIS:

The Diophantine equation representing the binary quadratic equation under consideration is

$$x^2 - a(t)y^2 - 2b(t)x + 2ya(t)c(t) = 0 \qquad [1]$$

Where a (t), b (t), c (t) are polynomials in Z[x].

The substitution of the transformations

$$x = u + b(t), y = v + c(t) \qquad [2]$$

In (1) leads to

$$u^2 = a(t)v^2 + b^2(t) - a(t)c^2(t) \qquad [3]$$

Whose initial solution is $v_0 = c(t), u_0 = b(t)$

To find the other solutions of (3), consider the Pellian equation

$$u^2 = a(t)v^2 + 1$$

Whose general solution $(\tilde{u}_n, \tilde{v}_n)$ is

$$\tilde{u}_n = \frac{1}{2}\left[(\tilde{u}_0 + \sqrt{a(t)}\tilde{v}_0)^{n+1} + (\tilde{u}_0 - \sqrt{a(t)}\tilde{v}_0)^{n+1}\right] = \frac{1}{2} f_n$$
$$\tilde{v}_n = \frac{1}{2\sqrt{a(t)}}\left[(\tilde{u}_0 + \sqrt{a(t)}\tilde{v}_0)^{n+1} - (\tilde{u}_0 - \sqrt{a(t)}\tilde{v}_0)^{n+1}\right] = \frac{1}{2\sqrt{a(t)}} g_n \quad [4]$$

Here $(\tilde{u}_0, \tilde{v}_0)$ is the initial solutions of the pellian.

Applying Brahmaguptha lemma between (u_0, v_0) and $(\tilde{u}_n, \tilde{v}_n)$, the solution of (3) are obtained from the relations

$$u_{n+1} = u_0\tilde{u}_n + a(t)v_0\tilde{v}_n = \frac{b(t)}{2} f_n + \frac{c(t)}{2}\sqrt{a(t)} g_n$$
$$v_{n+1} = v_0\tilde{u}_n + u_0\tilde{v}_n = \frac{c(t)}{2} f_n + \frac{b(t)}{2a(t)}\sqrt{a(t)} g_n \quad [5]$$

In view of [2] and [5], the corresponding non zero distinct integral solutions of [1] are given by

$$x_{n+1} = \frac{b(t)}{2} f_n + \frac{c(t)}{2}\sqrt{a(t)} g_n + b(t)$$
$$y_{n+1} = \frac{c(t)}{2} f_n + \frac{b(t)}{2a(t)}\sqrt{a(t)} g_n + c(t)$$

A few interesting properties and examples are presented below:

Example: 1

Take a=3, b=5 c=2

Hence, the corresponding non zero distinct integral solutions of (1) are given by

$$x_{n+1} = \frac{5}{2} f_n + \sqrt{3} g_n + 5$$
$$y_{n+1} = f_n + \frac{5}{6}\sqrt{3} g_n + 2$$

Where

$$f_n = [(2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1}]$$
$$g_n = [(2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}]$$

Example: 2

Let $a = t^2 + 2t, b = t+1, c = 1$

Hence, the corresponding non zero distinct integral solutions of (1) are found to be

$$x_{n+1} = \frac{1}{2} f_n + t + 1$$
$$y_{n+1} = \frac{1}{2\sqrt{t(t+2)}} g_n + 1$$

Where

$$f_n = [((t+1) + \sqrt{t(t+2)})^{n+1} + ((t+1) + \sqrt{t(t+2)})^{n+1}]$$
$$g_n = [((t+1) + \sqrt{t(t+2)})^{n+1} - ((t+1) + \sqrt{t(t+2)})^{n+1}]$$

Properties:

1. **Recurrence relation for ax:**

$$x_{n+3} - 2\tilde{u}_0 x_{n+2} - (a(t)\tilde{v}_0^2 - \tilde{u}_0^2)x_{n+1} = b(t)[1 - 2\tilde{u}_0 - (a(t)\tilde{v}_0^2 - \tilde{u}_0^2)]$$

Where $x_0 = 2b(t), x_1 = b(t)\tilde{u}_0 + a(t)c(t)\tilde{v}_0 + b(t)$

2. **Recurrence relation for y:**

$$y_{n+3} - 2\tilde{u}_0 y_{n+2} - (a(t)\tilde{v}_0^2 - \tilde{u}_0^2)y_{n+1} = c(t)[1 - 2\tilde{u}_0 - (a(t)\tilde{v}_0^2 - \tilde{u}_0^2)]$$

Where $y_0 = 2c(t), y_1 = c(t)\tilde{u}_0 + b(t)\tilde{v}_0 + c(t)$

3. $\dfrac{2\left\lfloor (b(t)\tilde{v}_0 + c(t)\tilde{u}_0)x_{2n+2} - c(t)x_{2n+3} - b(t)(b(t)\tilde{v}_0 + c(t)\tilde{u}_0 - c(t) \right\rfloor}{b(t)(b(t)\tilde{v}_0 + c(t)\tilde{u}_0) - c(t)(\tilde{u}_0 + c(t)a(t)\tilde{v}_0)} + 2$ **is a perfect square.**

4. $\frac{6\left(2bx_{2n+2}-2acy_{2n+2}\right)}{b^2-ac^2}$ is a nasty number.

5. $\frac{\left(2b(t)x_{3n+3}-2a(t)cy_{3n+3}+3(2b(t)x_{n+1}-2a(t)c(t)y_{n+1})-8(b(t)^2-a(t)c^2(t)\right)}{b^2(t)-a(t)c^2(t)}$

is a cubical integer.

6. $[2a(t)b(t)y_{n+1}-2a(t)c(t)x_{n+1}]^2 = a(t)\left(2b(t)x_{n+1}-2a(t)c(t)y_{n+1}-2(b^2(t)-a(t)c^2(t))^2\right)^2$
$-4a(t)(b^2(t)-a(t)c^2(t))$

is a hyperbola.

7. $2a(t)[b(t)y_{n+1}-c(t)x_{n+1}]^2 = \left(b(t)x_{2n+2}-a(t)c(t)y_{2n+2}\right)-2(b^2(t)-a(t)c^2(t))^2$

is a parabola.

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