Impact J. Sci. Tech., Vol. 14, No. 2, 1-14, 2020
ISSN - 0973-8290



# COMMON FIXED POINT THEOREMS IN SEQUENTIALLY COMPACT D*-METRIC SPACES

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doi:10.46598/impactjst.14.2.2020.300

URL: doi:10.46598/impactjst.14.2.2020.300

**In this paper, we give common fixed point theorems in sequentially compact D*-metric spaces for four or more mappings. We obtain a modification of a theorem of Dhage in D*-metric spaces.**

## 1. INTRODUCTION AND PRELIMINARIES

**Dhage [1] introduced the notion of generalized metric or D-metric as follows:**

**Let X be a nonempty set. A generalized metric (or D-metric) on X is a function**

**$D:X^3 \to [0, \infty)$ that satisfies the following conditions for each** $x, y, z, a \in X$,

(1)$D(x, y, z) = 0$ **if and only if** $x=y=z$,

(2)$D(x, y, z) = D(p\{x, y, z\})$, **where p is a permutation function, (symmetry)**

(3)$D(x, y, z) \le D(x, y, a) + D(x, a, z) + D(a, y, z)$ **(Tetrahedral inequality).**

**The pair** $(X, D)$ **is called a generalized metric space or a D-metric space.**

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**A sequence** $\{x_n\}$ **in** (X, D) **is said to be convergent to** x **in** X **if and only if** $D(x_n, x_m, x) \to 0$ as $n \to \infty$. $\{x_n\}$ **in** X is **called D- Cauchy sequence If D** $(x_n, x_m, x_p) \to 0$ as $n, m, p \to \infty$. **The D-metric space** (X, D) **is said to be complete**

**if every D-Cauchy sequence in** X **is convergent in** X.

**Let** (X, D) **be a D-metric space. For** $x_0$ in X **and** $r > 0$, the **set**

$B(x_0, r) = \{y \in X : D(x_0, y, y) < r\}$ **is called an open ball with center** $x_0$ **and radius** r**.**

**Dhage [1] claimed that D-metric convergence defines a Hausdorff topology and D-metric is sequentially continuous in all the three variables. Several authors have taken these claims for granted and used them in proving some fixed point theorems in D-metric spaces. Unfortunately, almost all theorems in D-metric spaces are either false or of doubtful validity (see [5,6,7]). Recently Sedghi, Rao and Shobe [4] and Sedghi and Shobe [3] introduced D*-metric space by modifying the tetrahedral inequality in D-metric space and proved some basic results in it, which are not true in D-metric space. In this paper we prove a common fixed point theorem for four maps in sequentially compact D*-metric spaces, which modify a theorem of Dhage [2]. We need some definitions, remarks and lemmas given in [3,4]. Let R $^+$ be the set of all non-negative real numbers and N be the set of all natural numbers.**

**DEFINITION 1.1: Let X be a non empty set. A generalized metric (or D*-metric) on X is a function:** $D^*: X^3 \to [0, \infty)$ **that satisfies the following conditions for each** x, y, z, a $\in$X,

(1) $D^*(x, y, z) = 0$ **if and only if** $x = y = z$,

(2) $D^*(x, y, z) = D^*(p\{x, y, z\})$, **(symmetry) where p is a permutation function,**

(3) $D^*(x, y, z) \le D^*(x, y, a) + D^*(a, z, z)$.

**The pair** $(X, D^*)$ **is called a generalized metric (or $D^*$-metric) space.**

**Immediate examples of such a function are**

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,

(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

**Here, d is the ordinary metric on X.**

(c) **if $X = R^n$ then we define**

$$D^*(x, y, z) = \left\{\|x-y\|^p + \|y-z\|^p + \|z-x\|^p\right\}^{\frac{1}{p}}$$

**for every** $p \in [1, \infty)$.

(d) **If** $X = R^+$ **then we define**

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise,} \end{cases}$$

**REMARK** 1.2: **In a** $D^*$**-metric space, we have** $D^*(x, x, y) = D^*(x, y, y)$.

**REMARK** 1.3: **Let** $(X, D^*)$ **be a** $D^*$**-metric space. If we define** $f : X^2 \to [0, \infty)$ **as** $f(x, y) = D^*(x, x, y)$ **for all** $x, y \in X$ **then** f **is an ordinary metric on** X.

**DEFINITION 1.4: Let** $(X, D^*)$ **be a $D^*$-metric space and** $A \subseteq X$.

(1) **For** $r > 0$ **define** $B_{D^*}(x, r) = \{y \in X: D^*(x, y, y) < r\}$.

(2) **If for every** $x \in A$ **there exists** $r > 0$ **such that** $B_{D^*}(x, r) \subset A$, **then**

**A is called open subset of** X.

(3) A is **said to be** $D^*$- **bounded if there exists** $r > 0$ **such that**

$D^*(x, y, y) < r$ for all $x, y \in A$.

(4) **A sequence** $\{x_n\}$ **in** X **convergence to** x if **and only if** $D^*(x_n, x_n, x) \to 0$ as $n \to \infty$.

(5) **A sequence** $\{x_n\}$ **in** X **is called a Cauchy sequence** $D^*(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. **The** $D^*$**-metric space** $(X, D^*)$ **is said to complete if every Cauchy sequence in** X **is convergent in** X.

**LEMMA 1.5: Let** $(X, D^*)$ **be a D*-metric space. If** $r > 0$, **then the ball** $B_{D^*}(x, r)$ **with center** $x \in X$ **and radius** r is **an open subset of** X.

**LEMMA 1.6: Let** $(X, D^*)$ **be a** $D^*$**-metric space. Then D* is a continuous function on** $X^3$.

**LEMMA 1.7: Let** $(X, D^*)$ **be a** $D^*$**-metric space. If sequence** $\{x_n\}$ **in** X **converges to** x, **then** x **is unique.**

**LEMMA 1.8: Let** $(X, D^*)$ **be a D*-metric space. If sequence** $\{x_n\}$ **in** X **converges to** x, **then sequence** $\{x_n\}$ **is a Cauchy sequence.**

**LEMMA 1.9:** $(X, D^*)$ **is said to be sequentially compact** $D^*$**-metric space if every sequence in** X **has a convergent sub sequence in it.**

**REMARK 1.10: Let f and S be two self maps on** $(X, D^*)$ **satisfying**

$$D^*(f x, f y, z) < \max\left\{\begin{array}{c} D^*(Sx, Sy, z), D^*(Sx, fx, z), D^*(Sy, fy, z), \\ D^*(Sx, fy, z), D^*(Sy, fx, z) \end{array}\right\}$$

**For every** $x, y, z \in X$ **for which the right hand side is not zero.**

**Then** $f = S$.

**PROOF: Suppose** $f x \neq Sx$ **for some** $x \in X$.

$$D^*(f x, f x, Sx) < \max\left\{\begin{array}{c} D^*(Sx, Sx, Sx), D^*(Sx, fx, Sx), D^*(Sx, fx, Sx), \\ D^*(Sx, fx, Sx), D^*(Sx, fx, Sx) \end{array}\right\}$$

$= D^*(fx, f x, Sx)$ **from REMARK** 1.2.

**It is a contradiction Hence** $f = S$.

**REMARK 1.11: If** $S = I$ **(Identity map) in REMARK 1.10 then** $f = I$. **Since the fixed point is unique, from inequality, it follows that** $X = a$ **singleton set.**

## 2.MAIN RESULTS

**THEOREM 2.1: Let f, g, S and T be self –mappings of a sequentially compact** $D^*$-**metric space** $(X, D^*)$ **such that** $f \neq g$,

(1) $f(X) \subseteq T(X)$ **and** $g(X) \subseteq S(X)$,

(2) $$D^*(f x, g y, z) < \max\left\{\begin{array}{c} D^*(Sx, Ty, z), D^*(fx, Sx, z), D^*(gy, Ty, z), \\ \frac{1}{2}[D^*(fx, Ty, z) + D^*(gy, Sx, z)] \end{array}\right\}$$

**For every** $x, y, z \in X$ **for which one of** $D^*(S x, T y, z)$, $D^*(f x, Sx, z)$

$D^*(gy, Ty, z)$ **is positive and** $z = Sx$ or $Ty$,

(3) **the pairs (f, S) and (g, T) are weakly compatible,**

(4) **f and S are continuous.**

**Then f, g, S and T have a unique common fixed point p in X. Further p is the unique common fixed point of f and S and of g and T.**

**PROOF: Let** $m = \inf\{D^*(fx, fx, Sx) : x \in X\}$

**Since f and S are continuous on sequentially compact D*-metric space, there exists** $u \in X$ **such that** $m = D^*(fu, fu, Su)$.

**Since** $f(X) \subseteq T(X)$, **there exists** $v \in X$ **such that** $fu = Tv$ ……… (5)

$D^*(Tv, Tv, Su) = D^*(fu, fu, Su) = m$.

**Suppose** $m > 0$. **Then from** (2) **and REMARK 1.2, we have**

$$D^*(fu, gv, Tv) < \max\left\{\begin{array}{c} D^*(Su, Tv, Tv), D^*(fu, Su, Tv), D^*(gv, Tv, Tv), \\ \frac{1}{2}[D^*(fu, Tv, Tv) + D^*(gv, Su, Tv)] \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} m, m, D^*(gv, Tv, Tv), \\ \frac{1}{2}[0 + D^*(Tv, Su, Tv) + D^*(Tv, gv, gv)] \end{array}\right\}$$

$$= \max\left\{m, m, D^*(gv, Tv, Tv), \frac{1}{2}[m + D^*(Tv, gv, gv)]\right\}$$

**If** $D^*(Tv, gv, gv) \geq m$ then $D^*(Tv, gv, gv) < D^*(Tv, gv, gv)$.

**Hence** $D^*(Tv, gv, gv) < m$ ………. (6)

**Since** $g(X) \subseteq S(X)$, **there exists** $w \in X$ **such that** $gv = Sw$ ……. (7)

**Now from** (2) **and REMARK 1.2, we have**

$$D^*(f w, g v, S w) < \max \left\{ \begin{array}{c} D^*(Sw, Tv, Sw), D^*(fw, Sw, Sw), D^*(gv, Tv, Sw), \\ \frac{1}{2}[D^*(fw, Tv, Sw) + D^*(gv, Sw, Sw)] \end{array} \right\}$$

$$\leq \max \left\{ \begin{array}{c} D^*(gv, Tv, gv), D^*(fw, Sw, Sw), D^*(gv, Tv, gv), \\ \frac{1}{2}[D^*(gv, Tv, Sw) + D^*(gv, fw, fw) + 0] \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} D^*(gv, Tv, gv), D^*(fw, Sw, Sw), D^*(gv, Tv, gv), \\ \frac{1}{2}[D^*(gv, Tv, gv) + D^*(Sw, fw, fw)] \end{array} \right\}$$

**If** $D^*(f w, S w, S w) \geq D^*(g v, T v, g v)$ then

$D^*(f w, S w, S w) < D^*(f w, S w, S w)$.

**Hence** $D^*(f w, S w, S w) < D^*(g v, T v, g v)$ ......... (8).

**From (8) and (6), we have** $m \leq D^*(f w, S w, S w) < D^*(g v, T v, g v) < m$.

**It is a contradiction. Hence** $m = 0$. **Thus**, $f u = S u$ ....... (9)

**Suppose** $T v \neq g v$. **Then** $D^*(g v, T v, g v) > 0$.

$$D^*(f u, g v, T v) < \max \left\{ \begin{array}{c} D^*(Su, Tv, Tv), D^*(f u, Su, Tv), D^*(gv, Tv, Tv), \\ \frac{1}{2}[D^*(f u, Tv, Tv) + D^*(gv, Su, Tv)] \end{array} \right\}$$

$$= \max \{0, 0, D^*(g v, T v, T v), \frac{1}{2} [0 + D^*(g v, T v, T v)]\}$$

$= D^*(g\,v, T\,v, T\,v)$.

**It is a contradiction. Hence** $g\,v = T\,v$ ……. (10)

**Thus**, $f\,u = S\,u = g\,v = T\,v = p$, say.

**Since the pair (f, S) is weakly compatible we have**

$f\,p = f\,f\,u = f\,Su = S\,f\,u = S\,Su = S\,p$ ……… (11)

**Suppose** $S\,p \neq p$. **Then** $D^*(p, S\,p, p) > 0$. From (2) we have

$$D^*(f\,p, g\,v, p) < \max\left\{\begin{matrix} D^*(Sp, Tv, p), D^*(f\ p, S\,p, p), D^*(gv, Tv, p), \\ \frac{1}{2}[D^*(f\ p, Tv, p) + D^*(gv, S\,p, p)] \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} D^*(fp, p, p), D^*(f\ p, fp,\ p), 0, \\ \frac{1}{2}[D^*(f\ p, p, p) + D^*(p, f\ p, p)] \end{matrix}\right\}$$

$= D^*(f\,p, p, p)$ from REMARK 1.2.

**It is a contradiction Thus,** nce $f\,p = p$.

**Thus**, $f\,p = S\,p = p$. ……… (12)

**Since the pair (g, T) is weakly compatible we have**

$g\,p = g\,g\,v = gT\,v = T\,g\,v = TTv = Tp$. ………. (13)

**Suppose** $T\,p \neq p$. Then $D^*(p, T\,p, T\,p) > 0$. **From (2) we have**

$$D^*(f\,u, g\,p, T\,p) < \max\begin{Bmatrix} D^*(Su, Tp, Tp), D^*(f\;u, S\,u, Tp), D^*(gp, Tp, Tp), \\ \frac{1}{2}[D^*(f\;u, Tp, Tp) + D^*(gp, S\,u, Tp)] \end{Bmatrix}$$

$$= \max\begin{Bmatrix} D^*(p, Tp, Tp), D^*(p, p, T\,p), 0, \\ \frac{1}{2}[D^*(p, Tp, Tp) + D^*(Tp, p, Tp)] \end{Bmatrix}$$

$= D^*(p, Tp, Tp)$.

**It is a contradiction. Hence** $T\,p = p$.

**Thus**, $g\,p = T\,p = p$.

**Thus, p is a common fixed point of** f, g, S **and** T.

**Suppose** $p_0$ **is another common fixed point** of f, g, S and T.

**Putting** $x = p$, $y = p_0$, $z = p_0$ in (2) **we get** $p = p_0$. **Thus**, p **is the unique common fixed point of** f, g, S **and** T.

**Suppose** $p_1$ **is another common fixed point of f and S. Then**

$D^*(p_1, p, p) = D^*(f\,p_1, g\,p, p)$

$$< \max\begin{Bmatrix} D^*(Sp_1, Tp, p), D^*(f\;p_1, S\,p_1, p), D^*(gp, Tp, Tp), \\ \frac{1}{2}[D^*(f\;p_1, Tp, p) + D^*(gp, S\,p_1, p)] \end{Bmatrix}$$

$$= \max\begin{Bmatrix} D^*(p_1, p, p), D^*(p_1, p_1, p), 0, \\ \frac{1}{2}[D^*(p_1, p, p) + D^*(p, p_1, p)] \end{Bmatrix}$$

$= D^*(p, p_1, p)$.

**It is a contradiction. Hence** $p_1 = p$.

**Thus, p is the unique common fixed point of f and S.**

**Similarly, we can show that p is the unique common fixed point of g and T.**

**REMARK 2.2: THEOREM 2.1with** $S = T = I$ **am a probable modification of Theorem 2.2 of [2]**.

**COROLLARY 2.3: THEOREM 2.1 with the inequality (2) is replaced by**

$[1+p\, D^*(Sx, Ty, z)]\, D^*(fx, gy, z)$

$$< p \max\left\{\begin{matrix} D^*(fx, Sx, z)\, D^*(gy, Ty, z), \\ D^*(fx, Ty, z)\, D^*(gy, Sx, z) \end{matrix}\right\}$$

$$+ \max\left\{\begin{matrix} D^*(Sx, Ty, z), D^*(f\ x, S\ x, z), D^*(gy, Ty, z), \\ \frac{1}{2}[D^*(f\ x, Ty, z) + D^*(gy, S\ x, z)] \end{matrix}\right\}$$

**COROLLARY 2.4: Let S, T and sequences** $\{f_i\}$, $\{g_j\}$ **for every** $i, j \in N$ **be self – mappings of a sequentially compact D*-metric space** $(X, D^*)$ **such that**

$f_i \neq g_j$ **for every** $i, j \in N$,

(1) **there exist** $i_0, j_0 \in N$ **such that** $f_{i_0}(X) \subseteq T(X), g_{j_0}(X) \subseteq S(X)$,

(2) $D^*(f_i x, g_j y, z) < \max \left\{ \begin{array}{l} D^*(Sx, Ty, z), D^*(f_i x, Sx, z), D^*(g_j y, Ty, z), \\ \frac{1}{2}[D^*(f_i x, Ty, z) + D^*(g_j y, Sx, z)] \end{array} \right\}$

**for every** $i, j \in N$ **and for every** $x, y, z \in X$ **for which one** of $D^*(Sx, Ty, z)$,

$D^*(f_i x, Sx, z)$, $D^*(g_j y, Ty, z)$ **is positive and** $z = Sx$ or $Ty$,

(3) **the pairs** $(S, f_{i_0})$ and $(g_{j_0}, T)$ **are weakly compatible,**

(4) $f_{i_0}$ **and S are continuous.**

**Then** S, T, $f_i$ **and** $g_j$ **have a unique common fixed point in** p **in** X **and further** p **is the unique common fixed point of** $f_i$ **and** S **and of** $g_j$ **and** T, **for every** i, j = 1, 2,

**PROOF: By THEOREM 2.1, the maps** $g_{j_0}, f_{i_0}$, S **and** T **have a unique common fixed point in** X. **That is, there exists a unique** $p \in X$ **such that** $g_{j_0}(p) = f_{i_0}(p) = S(p) = T(p) = p$.

**Let there exist** $j \in N$ **such that** $j \neq j_0$ **and** $D^*(g_j(p), p, p) > 0$. **Then by (2)**

**we have**

$$D^*(p, g_j p, p) = D^*(f_{i_0} p, g_j p, p)$$

$$< \max \left\{ \begin{array}{l} D^*(Sp, Tp, p), D^*(f_{i_0} p, Sp, p), D^*(g_j p, Tp, p), \\ \frac{1}{2}[D^*(f_{i_0} p, Tp, p) + D^*(g_j p, Sp, p)] \end{array} \right\}$$

$= \max\{0,0, D^*(g_j p, p, p), \frac{1}{2}[0+ D^*(g_j p, p, p)]\}$

$= D^*(g_j p, p, p)$.

**It is a contradiction. Hence for every** $j \in N$ **we have** $g_j(p) = p$. **Similarly, as in above, we can show that** $f_i(p) = p$ **for every** $i \in N$. **The rest of the proof follows as in THEOREM 2.1.**

**COROLLARY 2.5: Let f, g, T, S, R and H be self-mappings of a sequentially compact D*-metric space (X, D*) such that f ≠ g,**

(1) $f(X) \subseteq TR(X)$ and $g(X) \subseteq SH(X)$,

(2) $$D^*(fx, gy, z) < \max\left\{\begin{array}{l} D^*(SHx, TRy, z), D^*(fx, SHx, z), D^*(gy, TRy, z), \\ \frac{1}{2}[D^*(fx, TRy, z) + D^*(gy, SHx, z)] \end{array}\right\}$$

**for every** $x, y, z \in X$ **for which one of** $D^*(SHx, TRy, z)$, $D^*(fx, SHx, z)$, $D^*(gy, TRy, z)$ **is positive and** $z = SHx$ or $TRy$,

(3) **the pairs** (SH, f) **and** (g, TR) **are weakly compatible,**

(4) f **and** SH **are continuous and** fR = Rf, gH = Hg **and** TRH = HTR, SHR = RSH.

**Then** S, H, T, R, f **and** g **have a unique common fixed point in p in X.**

**PROOF: By THEOREM 2.1, the maps** g, f, SH **and** TR **have a unique common fixed point in** X. **That is there exists a unique** $p \in X$ **such that**

g(p) = f(p) = SH(p) = TR(p) = p. **Further p is the unique common fixed point of f and SH and of g and TR.**

**Now** H(p) = Hg(p) = gH(p) **and** H(p) = HTR(p) = TRH(p). **Hence H(p) is a common fixed point of g and TR. But p is the unique common fixed point of g and TR. Hence** H(p) = p. Also, S(p) = SH(p) = p. **Similarly**

R(p) = Rf(p) = fR(p) and R(p) = RSH(p) = SHR(p). **That is R(p) is a common fixed point of f and SH. Hence** R(p) = p. Also, T(p) = TR(p) = p. **uniqueness of common fixed point of** S, H, T, R, f **and** g **follows by inequality** (2).

**COROLLARY 2.6: Let** $f^n$ **and** $g^m$ **for some n, m** $\in$**N be self-mappings of a sequentially compact D*-metric space (X, D*) such that** $f^n \neq g^m$,

$$D^*(f^n x, g^m y, z) < \max \begin{Bmatrix} D^*(x, y, z), D^*(f^n x, x, z), D^*(g^m y, y, z), \\ \frac{1}{2}[D^*(f^n x, y, z) + D^*(g^m y, x, z)] \end{Bmatrix}$$

**For every** x, y, z $\in$X **for which one of** $D^*(x, y, z)$, $D^*(f^n x, x, z)$, $D^*(g^m y, y, z)$ **is positive and z = x or y, and** $f^n$ **is continuous.**

**Then f and g have a unique common fixed point of p in X and p is the unique fixed point of f and g.**

**PROOF: By THEOREM 2.1 with T, S = I, it follows that** $f^n$ **and** $g^m$ **have a unique common fixed point p** $\in$**X and p is the unique fixed point of** $f^n$ **and** $g^m$**. Now** f (p) = f ($f^n$(p)) = $f^n$ (f (p). **Thus f (p) is a fixed point of** $f^n$.

**Since p is the unique fixed point of** $f^n$**, it follows that f (p) = p, similarly we have g (p)** = p. **The rest of the proof follows as in THEOREM 2.1.**

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