ON FIVE SERIES EQUATION INVOLVING GENERALIZED BATEMAN-k FUNCTIONS

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ABSTRACT:

Formal solution of five series equations involving generalized Bateman k-functions is given in this paper. Finally the solution of five series equations is reduced to the solution of a Fredholm integral equation of the second kind.

1. INTRODUCTION:

On going through the literature closely, we see that most of it are confined to find the solution of integral and series equations upto only three or four. In this paper an attempt has been made to find the solution of five series equations involving generalized Bateman k-function. Our method is similar to that of Cooke [2] used in

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used in solving triple equations of Bessel function. The analysis given here is formal and no attempt is made to justify the various limiting process.

2. PRILIMINARY RESULTS:

In this section we list some results involving generalized Bateman k-functions.

The orthogonality relation* is

$$\int_0^\infty x^{-2\alpha-2\sigma-1} k_{2(m+\alpha)}^{2(\alpha+\sigma)}(x) k_{2(n+\alpha)}^{2(\alpha+\sigma)}(x)dx = \frac{2^{2\alpha+\sigma}\cdot\Gamma(n-\sigma)}{\Gamma(2\alpha+\sigma+n+1)}\delta_{mn} \quad [1]$$

Where δ_{mn} is a kronecker delta, $\alpha+\sigma+1>0, \alpha+1<0$. For $\alpha+\sigma>-1, \beta>0$, we obtain the following equations from Srivastava [1]

$$\int_0^\xi e^x(\xi-x)^{\beta-1} k_{2(n+\alpha)}^{2(\alpha+\sigma)}(x)dx = \frac{\Gamma(\beta)}{2^\beta} e^\xi k_{2(n+\alpha)+\beta}^{2(\alpha+\sigma)+\beta}(\xi) \quad [2]$$

and for $2\alpha+\sigma+n+1>\beta>0$, the equation

$$\int_\xi^\infty e^{-x} x^{-2\alpha-2\sigma-1}(x-\xi)^{\beta-1} k_{2(n+\alpha)}^{2(\alpha+\sigma)}(x)dx$$
$$= \frac{\Gamma(\beta)\Gamma(2\alpha-\beta+\sigma+n-1)}{\xi^{2\alpha\cdot\beta+2\sigma+1}\cdot\Gamma(2\alpha+\sigma+n+1)} e^{-\xi} k_{2(n+\alpha)-\beta}^{2(\alpha+\sigma)\cdot\beta}(\xi) \quad [3]$$

We deduce the value of the series

$$S(r,x) = \sum_{n=0}^{\infty} \frac{\Gamma(2v+\sigma+n+1)}{2^{2\beta+2\sigma}\Gamma(n-\sigma)} k_{2(n+\alpha)}^{2(\beta+\sigma)}(r) k_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) \quad [4]$$

From the following series given by Srivastava [1]

$$M(\xi,x) = \sum_{n=0}^{\infty} \frac{\Gamma(1-\lambda)\Gamma(2v+\sigma+n+1)}{2^{2\beta+2\sigma-\lambda}\Gamma(n-\sigma)} k_{2(n+\beta)-\lambda}^{2(\beta+\sigma)-\lambda}(\xi) k_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) \quad [5]$$

* Throughout this section σ will be understood to take on negative integral values.

Multiplying both the sides of equation [5] by $e^{\xi}(r-\xi)^{\lambda-1}$ and integrating between the limits 0 to r and using [2], we get

$$\int_0^r e^{\xi}(r-\xi)^{\lambda-1}M(\xi,x)d\xi = \Gamma(\lambda)\Gamma(1-\lambda)e^{r}S(r,x) \quad [6]$$

Substituting the value of $M\{\xi,x\}$ for $\lambda = 2\beta - 2v$ which was obtained from Srivastava [1]

$$M\{\xi,x) = \frac{\xi^{2v+2\sigma+1}\cdot\Gamma(2v-2\beta+1)}{2^{2v-2\alpha}\Gamma(2\alpha-2v)}e^{\xi-x}(x-\xi)^{2\alpha-2v-1}H(x-\xi)$$

Where, $H(t)$ denotes Heaviside's unit function,

$\alpha > v > \beta - \frac{1}{2}, \beta + \sigma + 1 > 0, v + \sigma + 1 > 0$ **and** $\sigma + 1 \le 0$

$$S(r,x) = \frac{e^{-x}2^{2\alpha-2v}}{\Gamma(2\alpha-2v)\Gamma(2\beta-2v)}\int_0^t E(\xi)(x-\xi)^{2\alpha-2v-1}(r-\xi)^{2\beta-2v-1}d\xi$$
$$= \frac{e^{-x}2^{2\alpha-2v}}{\Gamma(2\alpha-2v)\Gamma(2\beta-2v)}S_t(r,x) \quad [7]$$

Where, $E(\xi) = e^{2\xi}\xi^{2v+2\sigma+1}, t = \min(x,r)$.

3. SOLUTION OF FIVE SERIES EQUATION:

We shall now solve five series equations involving generalized Bateman k-function, defined as below:

$$\sum_{n=0}^{\infty}\frac{A_n}{\Gamma(2v+\sigma+n+1)}k_{2(\beta+1)}^{2(\beta+\sigma)}(x) = 0; 0 < x < a \quad [8]$$

$$\sum_{n=0}^{\infty}\frac{A_n(1+H_n)}{\Gamma(2\beta+\sigma+n+1)}k_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) = f(x); a < x < b \quad [9]$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2v+\sigma+n+1)} k_{2(\beta+1)}^{2(\beta+\sigma)}(x) = 0; b < x < c \quad [10]$$

$$\sum_{n=0}^{\infty} \frac{A_n(1+H_n)}{\Gamma(2\beta+\sigma+n+1)} k_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) = g(x); c < x < d \quad [11]$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2v+\sigma+n+1)} k_{2(\beta+1)}^{2(\beta+\sigma)}(x) = 0; d < x < \infty \quad [12]$$

To solve equations [8] to [12] we assume that

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2v+\sigma+n+1)} k_{2(\beta+n)}^{2(\beta+\sigma)}(x) = \begin{cases} \phi_1(x); a < x < b \\ \phi_2(x); c < x < d \end{cases} \quad [13]$$

and on using the orthogonality relation it follows from equation [8], [10], [12] and [13]

$$A_n = \frac{\Gamma(2v+\sigma+n+1)\Gamma(2\beta+v+n+1)}{2^{2\beta+2\sigma}\Gamma(n-\sigma)} \left\{ \int_a^b \phi_1(r) + \int_c^d \phi_2(r) \right\} \cdot r^{-2\beta-2\sigma-1} k_{2(n+\beta)}^{2(\beta+\sigma)}(r)\,dr \quad [14]$$

On substituting for A_n **in equation [9] and [11], interchanging the order of summation and integration, we find**

$$\int_a^x r^{-2\beta-2\sigma-1}\phi_1(r) S_r(r,x)\,dr + \int_x^b r^{-2\beta-2\sigma-1}\phi_1(r) S_x(r,x)\,dr + \int_c^d r^{-2\beta-2\sigma-1}\phi_2(r) S_x(r,x)\,dr + \int_a^b r^{-2\beta-2\sigma-1}\phi_1(r) T(r,x)\,dr + \int_c^d r^{-2\beta-2\sigma-1}\phi_2(r) T(r,x)\,dr = \frac{\Gamma(2\alpha-2v)\Gamma(2\beta-2v)}{2^{2\alpha-2v}e^{-x}} f(x) \quad [15]$$

$$(a < x < b)$$

$$\int_a^b r^{-2\beta-2\sigma-1}\phi_1(r)S_r(r,x)dr + \int_c^x r^{-2\beta-2\sigma-1}\phi_2(r)S_r(r,x)dr$$
$$+\int_x^d r^{-2\beta-2\sigma-1}\phi_2(r)S_x(r,x)dr + \int_a^b r^{-2\beta-2\sigma-1}\phi_1(r)T(r,x)dr + \quad [16]$$
$$\int_c^d r^{-2\beta-2\sigma-1}\phi_2(r)T(r,x)dr = \frac{\Gamma(2\alpha-2v)\Gamma(2\beta-2v)}{2^{2\alpha-2v}}e^{-x}g(x)$$
$$(c < x < d)$$

Where,

$$T(r,x) = \sum_{n=0}^{\infty} \frac{\Gamma(2v+\sigma+n+1)}{2^{2\beta+2\sigma}\Gamma(n-\sigma)} H_n k_{2(n+\alpha)}^{2(\beta+\sigma)}(r) k_{2(n+\alpha)}^{2(\alpha+\sigma)}(x) \quad [17]$$

Inverting the order of integration in the above equation, we find

$$\int_a^x \frac{E(\xi)}{(x-\xi)^{1-2\alpha+2v}}\{p_1(\xi) + \int_c^d \frac{r^{-2\beta-2\sigma-1}\phi_2(r)}{(r-\xi)^{1-2\beta+2v}}dr\}d\xi$$
$$\int_a^b r^{-2\beta-2\sigma-1}\phi_1(r)T(r,x)dr + \int_c^d r^{-2\beta-2\sigma-1}\phi_2(r)T(r,x)dr \quad [18]$$
$$= \frac{\Gamma(2\alpha-2v)\Gamma(2\beta-2v)}{2^{2\alpha-2v}e^{-x}}f(x);\ a < x < b$$

$$\int_c^x \frac{E(\xi)}{(x-\xi)^{1-2\alpha+2v}}q_1(\xi)d\xi + \{\int_a^b r^{-2\beta-2\sigma-1}\phi_1(r) + \int_c^d r^{-2\beta-2\sigma-1}\phi_2(r)\}T(r,x)dr$$
$$= \frac{\Gamma(2\alpha-2v)\Gamma(2\beta-2v)}{2^{2\alpha-2v}e^{-x}}g(x) - \int_a^b \frac{E(\xi)}{(x-\xi)^{1+2\alpha-2v}}p_1(\xi)d\xi \quad [19]$$
$$-\int_0^c \frac{E(\xi)}{(x-\xi)^{1-2\alpha+2v}}d\xi \cdot \int_c^d \frac{r^{-2\beta-2\alpha-1}\phi_2(r)}{(r-\xi)^{1-2\beta+2v}}dr;\ c < x < d$$

And

$$\text{(i)}\,p_1(\xi) = \int_\xi^b \frac{r^{-2\beta-2\sigma-1}\phi_1(r)}{(r-\xi)^{1-2\beta+2v}}dr \quad [20]$$

$$\text{(ii)}\,q_1(\xi) = \int_\xi^d \frac{r^{-2\beta-2\sigma-1}\phi_1(r)}{(r-\xi)^{1-2\beta+2v}}dr$$

For $< 1 - 2\beta + 2v < 1, 0 < 1 - 2\alpha + 2v < 1$, we can solve Abel type integral equations [18], [19] and [20] to obtain the equations

$$E(\xi)p_1(\xi) - \int_a^b p_1(t)M(\xi,t)dt = G(\xi) - \int_c^d q_1(t)N(\xi,t)dt, a < \xi < b \quad [21]$$

And

$$E(\xi)q_1(\xi) - \int_c^d q_1(t)N_1(\xi,t)dt = H(\xi) - \int_a^b p_1(t)M_1(\xi,t)dt, c < \xi < d \quad [22]$$

Where

$$G(\xi) = \frac{\Gamma(2\beta - 2v)}{2^{2\alpha-2v}\Gamma(1 - 2\alpha + 2v)} \frac{d}{d\xi} \int_a^\xi \frac{e^x f(x)}{(\xi - x)^{2\alpha-2v}} dx, a < \xi < b; \quad [23]$$

$$H(\xi) = \frac{\Gamma(2\beta - 2v)}{2^{2\alpha-2v}\Gamma(1 - 2\alpha + 2v)} \frac{d}{d\xi} \int_c^d \frac{e^x g(x)}{(\xi - x)^{2\alpha-2v}} dx, c < \xi < d, \quad [24]$$

$$M(\xi,t) = \frac{\sin(1 - 2\alpha + 2v)\pi \sin(1 - 2\beta + 2v)\pi}{\pi^2(2\beta - 2v)^{-1}} \frac{d}{d\xi} \int_a^\xi \frac{U(t,x)}{(\xi - x)^{2\alpha-2v}} dx, \quad [25]$$

$$N(\xi,t) = \frac{\sin(1 - 2\alpha + 2v)\pi E(\xi)(t - c)^{2v-2\beta}}{\pi(t - \xi)(c - \xi)^{2v-2\beta}}$$

$$- \frac{\sin(1 - 2\alpha + 2v)\pi \sin(1 - 2\beta + 2v)\pi}{\pi^2(2\beta - 2v)^{-1}} \frac{d}{d\xi} \int_a^\xi \frac{V(t,x)}{(\xi - x)^{2\alpha-2v}} dx \quad [26]$$

$$M_1(\xi,t) = \frac{\sin(1 - 2\alpha + 2v)\pi E(t)(c - t)^{2\alpha-2v}}{\pi(\xi - t)(\xi - c)^{2\alpha-2v}}$$

$$- \frac{\sin(1 - 2\alpha + 2v)\pi \sin(1 - 2\beta + 2v)\pi}{\pi^2(2\beta - 2v)^{-1}} \frac{d}{d\xi} \int_c^\xi \frac{U(t,x)}{(\xi - x)^{2\alpha-2v}} dx \quad [27]$$

$$N_1(\xi,t) = \frac{\sin(1 - 2\alpha + 2v)\pi \sin(1 - 2\beta + 2v)\pi}{\pi^2}$$

$$[(2\beta - 2v)\frac{d}{d\xi_c}\int \frac{V(t,x)dx}{(\xi - x)^{2\alpha-2v}} \cdot \frac{(\xi - c)^{2v-2\alpha}}{(t - c)^{2\beta-2v}} \cdot \int_a^c \frac{E(t)(c - t)^{2\alpha+2\beta-4v}}{(\xi - t)(x - t)} dt] \quad [28]$$

$$U(t,x) = \int_0^t \frac{T(r,x)dr}{(t-r)^{1+2\beta-2v}} \quad [29]$$

$$V(t,x) = \int_c^t \frac{T(r,x)dr}{(t-r)^{1+2\beta-2v}} \quad [30]$$

Once $p_1(\xi)$ and $q_1(\xi)$ is determined from equation [20], $\phi_1(r)$ and $\phi_2(r)$ are then found from the following equation:

$$r^{-2\beta-2\sigma-1}\phi_1(r) = -\frac{\sin(1-2\beta+2v)\pi}{\pi}\frac{d}{dr}\int_r^b \frac{p_1(\xi)d\xi}{(\xi-r)^{2\beta-2v}}, a < r < b \quad [31]$$

$$r^{-2\beta-2\sigma-1}\phi_2(r) = -\frac{\sin(1-2\beta+2v)\pi}{\pi}\frac{d}{dr}\int_r^d \frac{q_1(\xi)d\xi}{(\xi-r)^{2\beta-2v}}, c < r < d \quad [32]$$

Finally the coefficient A_n satisfying equation [8] to [12] are given by equation [14]. In particular, if we take $H_n = 0$ in equation [21] and [22], we obtain the results of Narain and Lal [3].

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