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TRIPLE TRIGNOMETRICAL INTEGRAL EQUATION

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ABSTRACT

An exact solution of triple trignometrical equations is obtained by using the finite Hilbert transform technique [4].

1. INTRODUCTION

In this paper, we formally solve the following triple integral equations

$$\int_0^\infty \ldots \int_0^\infty \prod_{k=1}^{r} A(\tau_k) \cot h\pi\tau_k \sin \tau_k\alpha_k d\tau_k = F_1(\alpha_1, \ldots, \alpha_r), \quad [1]$$

$$(0 < \alpha_\kappa < \alpha_\kappa);$$

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$$\int_0^\infty \dots \int_0^\infty \prod_{k=1}^{r} A(\tau_k)\sin\tau_k\,\alpha_k\,d\tau_k = F_2(\alpha_1,\dots,\alpha_r), \qquad [2]$$

$$(0 < \alpha_\kappa < b_\kappa);$$

$$\int_0^\infty \dots \int_0^\infty \prod_{k=1}^{r} A(\tau_k)\cot h\pi\tau_k \sin\tau_k\alpha_k d\tau_k = F_3(\alpha_1,\dots,\alpha_r), \qquad [3]$$

$$(\alpha_\kappa < b_\kappa);$$

$$K=1,2,3,\dots\dots\dots, r.$$

The functions $F_1(\alpha_1,....,\alpha_r)$, $F_2(\alpha_1,....,\alpha_r)$ and $F_3(\alpha_1,....,\alpha_r)$ are prescribed. These integral equations are to be solved for the unknown $A(\tau_k)$.

Here the exact solution of the integral equations (1), (2) and (3) has been obtained by using the finite Hilbert transform technique, developed by Srivastava and Lowengrub [4] for the solution of triple integral equation.

The analysis given here is purely formal and no attempt is made to justify the various limiting process. Tricomi [1] has discussed a theorem for finite Hilbert transforms, but here we shall use the modified Hilbert transform theorem.

2. THE MODIFIED HILBERT TRANSFORM THEOREM

If $p \in L_2(a,b)$, then the integral equation

$$F_0[g(\cos h\,x)] = \frac{1}{\pi}\int_a^b \frac{g(\cos h\,x)\sin h\,x dx}{\cos h\,x - \cos h\,y} = p(y), y \in (a,b); \qquad [4]$$

has the solution

$$F_y^{-1}[p(y)] = g(\cosh x)$$

$$= -\frac{1}{\pi}\left(\frac{\cos h\,x - \cos h\,a}{\cos h\,b - \cos h\,x}\right)^{\frac{1}{2}} \int_a^b \left(\frac{\cos h\,b - \cos h\,y}{\cos h\,y - \cos h\,a}\right)^{\frac{1}{2}}.$$

$$\frac{p(y)\sin h\, y dy}{\cos h\, y - \cos h\, x} + \frac{C}{[(\cos h\, x - \cos h\, a)(\cos h\, b - \cos h\, x)]^{\frac{1}{2}}} \qquad [5]$$

Where C is an arbitrary constant and the first term belongs to the class L_2 (a,b). (F_y is called the finite Hilbert transform).

3. SOLUTION OF TRIPLE INTEGRAL EQUATIONS

In this section, we shall consider the triple integral equations [1], [2] and [3]. Let us suppose that

$$\int_0^\infty \dots \int_0^\infty \prod_{k=1}^{r} A(\tau_k)\cot\ h\pi\tau_k \sin\ \tau_k\alpha_k d\tau_k = g[\cosh\ (\alpha_1, \dots, \alpha_r)]\cosh\ \frac{1}{2}\alpha_k \qquad [6]$$

$$(a_k < \alpha_k < b_k);$$

Using the inversion theorem for the Fourier transform and the relations [1], [6] and [3], we have

$$\prod_{k=1}^{r} A(\tau_k)\cot\ h\pi\tau_k = \prod_{k=1}^{r} [(\frac{2}{\pi}) \int_0^{r_a} F_1(s_1, \dots \dots s_r)\sin\ \tau_k s_k ds_k$$

$$+(\frac{2}{\pi})^r_{a_k} \int_{g_k}^{b_k} g[\cosh\ (s_1, \dots \dots, s_r)]\sin\ \tau_k s_k \cosh\ \frac{1}{2} s_k ds_k \qquad [7]$$

$$+\left(\frac{2}{\pi}\right)^r \int_{b_k}^{\infty} F_3(s_1, \dots \dots, s_r)]\sin\ \tau_k s_k ds_k]$$

Substituting the value of $A(\tau_k)$ from [7] in [2], interchanging the order of integration and using the result

$$\int_0^\infty \tau^{-1}\tanh\ \pi\tau \sin\tau s \sin\tau\alpha\, d\tau = \frac{1}{2}\log\left|\frac{\sinh\frac{1}{2}s + \sinh\frac{1}{2}\alpha}{\sinh\frac{1}{2}s - \sin\frac{1}{2}\alpha}\right| \qquad [8]$$

Which can be easily obtained from (5, p. 516, 4.116 (2)], we find that

$$\prod_{k=1}^{r}\int_{a_k}^{b_k} g[\cosh(s_1,\dots\dots,s_r)]\cosh\frac{1}{2}s_k\cdot\log\left|\frac{\sinh\frac{1}{2}s_k + \sinh\frac{1}{2}\alpha_k}{\sinh\frac{1}{2}s_k - \sin\frac{1}{2}\alpha_k}\right|ds_k \qquad [9]$$

$$= R(\alpha_1,\dots,\alpha_r) \qquad (a_k < \alpha_k, b_k)$$

k = 1, 2, 3, ..., r.

Where

$$R(\alpha_1,\dots,\alpha_r) = \pi F_2(\alpha_1,\dots,\alpha_r)$$

$$\prod_{k=1}^{r}\left[\int_0^{a_k} F_1(s_1,\dots,s_r)\log\left|\frac{\sinh\frac{1}{2}s_k + \sinh\frac{1}{2}\alpha_k}{\sinh\frac{1}{2}s_k - \sin\frac{1}{2}\alpha_k}\right|ds_k + \right. \qquad [10]$$

$$\int_{b_k}^{\infty} F_3(s_1,\dots,s_r)\log\left|\frac{\sinh\frac{1}{2}s_k + \sinh\frac{1}{2}\alpha_k}{\sinh\frac{1}{2}s_k - \sin\frac{1}{2}\alpha_k}\right|ds_k$$

Differentiating (9), we see that g must also be a solution of the integral equation

$$\prod_{k=1}^{r}\left(\frac{1}{\pi}\right)r\int_{a_k}^{b_k}\frac{g[\cosh(s_1,\dots,s_r)]\sinh\ s_k}{(\cosh\ s_k - \cosh\ \alpha_k)}ds_k = \frac{R'(\alpha_1,\dots,\alpha_r)}{\pi^r\cosh\frac{1}{2}\alpha_k} \qquad [11]$$

$$(a_k < \alpha_k < b_k),$$

Hence, an using modified Hilbert transform technique [eq. (4),(5) and eq. (11), (5)], the function g is given by

$$g[\cosh(s_1, \ldots, s_r)] = \prod_{k=1}^{r} \left(-\frac{2}{\pi^2}\right)\left(\frac{\cosh s_k - \cosh a_k}{\cosh b_k - \cosh s_k}\right)^{\frac{1}{2}}$$

$$\int_{a_k}^{b_k} \left(\frac{\cosh b_k - \cosh \alpha_k}{\cosh \alpha_k - \cosh \alpha_k}\right)^{\frac{1}{2}} \cdot \frac{R'(\alpha_1, \ldots, \alpha_r)\sinh \frac{1}{2}\alpha_k d\alpha_k}{(\cosh \alpha_k - \cos s_k)} \qquad [12]$$

$$+ \frac{C}{[(\cosh s_k - \cosh a_k)(\cosh b_k - \cosh s_k)]^{\frac{1}{2}}}$$

Where, C is an arbitrary constant. The special case of most important is the one in which $F_1 = F_3 = 0$, $F_2 = 1$. In this case, we get

$$g[\cosh(s_1, \ldots, s_r)] = \prod_{k=1}^{r} \frac{C}{[(\cosh s_k - \cosh a_k)(\cosh b_k - \cosh s_k)]^{1/2}} \qquad [13]$$

REFERENCES

[1] F.G. Tricomi, On the finite Hilbert transformations, Quart. J. Math. Oxford Ser (2) 2 (1951), 199 – 211.

[2] C.J. Tranter, Some triple integral equations, Proc. Glasgow Math. Assoc. 4 (1960), 200 – 203.

[3] I.N. Sneddon, Mixed boundary value problem in potential theory (Amsterdam, 1966).

[4] K.N. Srivastava and M. Lowengrub, Finite Hilbert transform technique for triple integral equations with trigonometric kernels, Proc. Roy. Soc. Edinburgh Sect. A68 (1970), 309 – 321.

[5] I.S. Gradsheyn and I.M. Ryzhik, Tables of integrals, series and products (Academic Press, 1965).