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Five Series Equations Involving Heat Polynomials

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Abstract:

An exact solution of the five series equations involving heat polynomials $P_{n,v}(x, t)$ is obtained in this paper. We have also shown the solution of the five series equations involving generalized Laguerre polynomials as a special case of the equations considered in the present paper.

Introduction:

In this paper, we consider the following five series equations.

$$\sum_{n=0}^{\infty} \frac{A_n t^{-n} \tilde{n}^n}{\Gamma(v+\frac{1}{2}+n+p)} P_{n+p,v}(x,-t) = f(x,t); 0 \leq x < y \quad (1.1)$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\mu+\frac{1}{2}+n+p)} P_{n+p,\mu}(\xi,-\tilde{n}) = \phi(x,t); y < x < z \quad (1.2)$$

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$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\mu+\frac{1}{2}+n+p)} P_{n+p,\mu}(\xi,-\tilde{n}) = \psi(x,t); z < x < h \tag{1.3}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\mu+\frac{1}{2}+n+p)} P_{n+p,\mu}(\xi,-\tilde{n}) = M(x,t); h < x < 1 \tag{1.4}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\mu+\frac{1}{2}+n+p)} P_{n+p,\sigma}(x,-t) = g(x,t); 1 < x < \infty \tag{1.5}$$

Where, f(x, t), ϕ(x, t), Ψ(x, t), M(x, t) and g(x, t) are prescribed functions for $t \geq \varrho > 0$ and An is to be determined and $P_{n,v}(x, t)$ is the heat polynomial (Haimo, 1996) defined by

$$P_{n,v}(x,t) = \sum_{k=0}^{n} 2^{2k}\binom{n}{k}\frac{\Gamma\left(v+\frac{1}{2}+n\right)}{\Gamma\left(v+\frac{1}{2}+n-k\right)} x^{2n-2k}t^{k}, v > 0 \tag{1.6}$$

It may be noted that is the ordinary heat polynomial of even order defined by Roserbloom and Widder (1959) and that The Hermite polynomial of even order defined by Erdelyi (1953).

We also define $W_{n,v}(x,t) = t^{-2n}G_v(x,t)P_{n,v}(x,-t) > 0$ **(1.7)**

Where, $G_v(x,t) = (2t)^{-v-\frac{1}{2}} \exp\left(-\frac{x^2}{4t}\right)$ **(1.8)**

And $W_{n,v}(x,t)$ **is the Appel transform of** $P_{n,v}(x,-t)$

The analysis given here is purely formal and no attempt is made to supply details of rigorous proof.

2. Certain integral and Series Representations

The heat polynomial $P_{n,v}(x,t)$is related to the generalized Laguerre polynomial by

$$P_{n,v}(x,-t) = (-1)^n\, 2^{2n} n!\, t^n h_n^{v-\frac{1}{2}}(\frac{x^2}{4t}) \qquad (2.1)$$

Using the orthogonality relation for $L_n^{\alpha}(x)$, it can easily be verified that for t > 0

$$\int_0^{\infty} W_{m,v}(x,t) P_{n,v}(x,-t)\, dR(x) = \delta_{m,n} k_n \qquad (2.2)$$

Where

$$dR\,(x) = 2^{\frac{1}{2}-v}[\Gamma(v+\frac{1}{2})]^{-1} x^{2v} dx \qquad (2.3)$$

And

$$k_n = \frac{\Gamma(v+\frac{1}{2})}{2^{4n} n!\, \Gamma(v+\frac{1}{2}+n)} \qquad (2.4)$$

Now, using the formula (27) of [1, p.190] in the form

$$(\frac{d}{dx})^m \{x^{\alpha+m} L_n^{(\alpha+m)}(x)\} = \frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)} x^{\alpha} L_n^{(\alpha)}(x) \qquad (2.5)$$

and from the relation (2.1), we obtain at once that

$$(\frac{d^2}{dx^2})^m \{x^{2v+2m-1}\, P_{n,v+m}(x,-t)\} = \frac{\Gamma(v+\frac{1}{2}+m+n)}{\Gamma(v+\frac{1}{2}+n)} x^{2v-1} p_{n,v}(x,-t) \qquad (2.6)$$

The relation

$$e^{-x_{1}(\alpha)}(x) = (-1)^m (\frac{d}{dx})^m \{e^{-x} L_n^{(a-m)}(x)\}, x \geq 0 \qquad (2.7)$$

together with (2.1) yields

$$(-4t)^m \left(\frac{d^2}{dx^2}\right)^m \{e^{-\frac{x^2}{4t}} D P_{n,v}(x,-t)\} = e^{-\left(\frac{x^2}{4t}\right)} P_{n,v}(x,-t) \quad (2.8)$$

Now we derive a few fractional integral type representations for $P_{n,v}(x,-t)$ and $W_{n,v}(x,t)$. Using the definition of Beta function and integrating the series for $P_{n,v}(x,-t)$ by term with respect to x, it can easily be seen that

$$P_{n,v+\beta}(\xi_1 - t) = 2\xi^{-2v-2\beta+1} \frac{\Gamma\left(\beta + v + \frac{1}{2} + n\right)}{\Gamma(\beta)\Gamma\left(v + \frac{1}{2} + n\right)} \int_0^{\xi} x^{2v}(\xi^2 - x^2)^{\beta-1} P_{n,v}(x,-t)dx, \quad (2.9)$$

$$(\beta > 0, v > -\frac{1}{2})$$

Using the following form of the Beta function formula

$$\int_0^{\infty} v^{-\lambda-s}(v-\sigma)^{\lambda-\mu-1}dv = \frac{\Gamma(\lambda-\mu)\Gamma(\mu+S)}{(\lambda+S)} \sigma^{-S-\mu} \quad (2.10)$$

Where $\lambda > \mu$ and $S + \mu > 0$ and integrating the series for $P_{n,v}(x,-t)$ term by term with respect to t, we get

$$\int_{\sigma}^{\infty} t^{-n-\mu-1}(t-\sigma)^{\mu-v-1} P_{n,v}(x,-t)dt = \frac{\Gamma(\mu-v)\Gamma\left(v+\frac{1}{2}+n\right)}{\Gamma\left(\mu+\frac{1}{2}+n\right)} \sigma^{-\left(v+\frac{1}{2}+n\right)} P_{n,\mu} \quad (2.11)$$

$$(x_t - \sigma); (\mu > v > -\frac{1}{2})$$

Expressing $P_{n,v}(x,-t)$ in terms of generalized Laguerre polynomial by means of (2.1) and using the formula [2, p, 405].

$$\int_{\xi}^{\infty} e^{-x}(x-\xi)^{\beta-1} L_n^{(\alpha)}(x)dx = \Gamma(\beta)e^{-\xi} L_n^{(\alpha-\beta)}(\xi), (\alpha+1 > \beta > 0) \quad (2.12)$$

It can be proved that

$$P_{n,v-\beta}(\xi,t) = 2^{1-\beta}\frac{t^{-\beta}}{\Gamma(\beta)}e^{(\frac{\xi^2}{4t})}\int_{\xi}^{\infty} x(x^2-\xi^2)^{\beta-1}w_{n,v}(x,t)dx, (v+\frac{1}{2}>\beta>0) \quad (2.13)$$

Now we derive certain series representation for $P_{n,v}(x,t)$. Using the generating relation to Haimo (1966)

$$(1-4zt)^{-v-\frac{1}{2}}\frac{e^{x^2z}}{(1-4zt)} = \sum_{n=0}^{\infty}\frac{z^n}{n!}P_{n,v}(x,t) \quad (2.14)$$

and the relation

$$(1-4zt)^{-v-\frac{1}{2}}\frac{e^{x^2z}}{(1-4zt)^{-(v-\mu)}}(1-4zt)^{-\frac{1}{2}-\mu}\frac{e^{x^2z}}{(1-4zt)}$$

It follows that

$$P_{n,v}(x,t) = \frac{\Gamma(n+1)}{\Gamma(v-\mu)}\sum_{k=0}^{n}\frac{\Gamma(v-\mu+n-k)(4t)^{n-k}}{\Gamma(n-k+1)\Gamma(k+1)}P_{k,\mu}(x,t) \quad (2.15)$$

Equation (2.15) can be inverted to get

$$2^{\frac{1}{2}-\mu}x^{2u}t^{-2k}G_{\mu}(x,t)P_{k,\mu}(x,-t) = \sum_{n=k}^{\infty}\frac{\Gamma(v-\mu+n-k)\Gamma(\mu+\frac{1}{2}+k)}{\Gamma(v-\mu)\Gamma(n-k+1)\Gamma(v+\frac{1}{2}+n)}$$

$$(-\frac{t}{4})^{n-k}2^{\frac{1}{2}-v}x^{2v}t^{-2n}G_{v}(x,t)P_{n,v}(x,-t), (2\mu>v) \quad (2.16)$$

Equation (2.16) follows from (2.15), since they each are equivalent to

$$\int_0^\infty 2^{\frac{1}{2}-v}x^{2v}t^{-2k}G_\mu(x,t)P_{k,\mu}(x,-t)P_{n,v}(x,-t)dx = \frac{\Gamma(v-\mu+n-k)\Gamma(\mu+\frac{1}{2}+k)}{\Gamma(v-\mu)\Gamma(n-k+1)}(-t)^{n-k}4^{n+k}n! \quad (2.17)$$

Solution of Five Series Equations:

Multiplying eqn. (1.1) by $t^{-(p+m+\mu+\frac{1}{2})}(t-\alpha)^{\mu+m-v-1}$ where m is a positive integer, integrating with respect to t from ϱ to ∞ and using (2.11), we get

$$\sum_{n=0}^{\infty}\frac{A_n}{\Gamma(\mu+m+n+p+\frac{1}{2})}P_{n+p,\mu+m}(\xi,-p)=F(\xi,p), 0<\xi<y \quad (3.1)$$

Where, $\mu+m>v-\frac{1}{2}$ and

$$F(\xi,p)=\frac{\tilde{n}^{v+\frac{1}{2}}}{\Gamma(\mu+m-v)}\int_i^\infty t^{-(p+m+\mu+\frac{1}{2})}(t-\tilde{n})^{\mu+m-v-1}f(\xi,t)dt \quad (3.2)$$

Further setting $t=\tilde{n}$ in eq. (1.5), multiplying it by $x(x^2-\xi^2)^{\sigma-\mu-1}, e^{-\frac{x^2}{4}}$, integrating with respect to x from ξ to ∞ and applying (2.13) we get

$$\sum_{n=0}^{\infty}\frac{A_n}{\Gamma(\mu+n+\tilde{n}+\frac{1}{2})^{P_{n+p+\mu}}(\xi,-\tilde{n})=G(\xi,\tilde{n}); y<\xi<\infty} \quad (3.3)$$

Where, $s>m>-\frac{1}{2}$ and

$$G(\xi,\rho)=2^{1-2(\sigma-\mu)}\tilde{\Pi}^{-(\sigma-\mu)}\frac{e^{(\frac{\xi^2}{4\tilde{n}})}}{\Gamma(\sigma-\mu)}\int_\xi^\infty x(x^2-\xi^2)^{\sigma-\mu-1}e^{-(\frac{x^2}{4\tilde{n}})}g(x,\xi)dx \quad (3.4)$$

Now, multiplying (3.1) by $\xi^{2(\mu+m)-1}$ and applying the operator$(\frac{d^2}{d\xi^2})^m$, we get in view of (2.6) that

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\mu+\frac{1}{2}+n+p)} P_{n+p,\mu}(\xi,-\tilde{n}) = H(\xi,\tilde{n}), 0<\xi<y \tag{3.5}$$

where,

$$H(\xi,\tilde{n}) = \xi^{1-2\mu}(\frac{d^2}{d\xi^2})^m[\xi^{2(\mu+m)-1}F(\xi,\hat{n})] \tag{3.6}$$

The left hand sides of (3.5), (1.2), (1.3), (1.4) and (3.3) are now identical and an application of the orthogonality relation (2.2) yields the solution of the equations (1.1), (1.2), (1.3), (1.4) and (1.5) in the form

$$A_n = \frac{\Gamma\left(\mu+\frac{1}{2}\right)}{2^{4(n+p)}(n+p)!} \left\{ \begin{array}{l} \int_0^y W_{n+p,\mu}\left(\xi,\tilde{\Pi}\right) H\left(\xi,\tilde{\Pi}\right) dR(\xi) + \\ \int_y^z W_{n+p,\mu}\left(\xi,\tilde{\Pi}\right) \phi\left(\xi,\tilde{\Pi}\right) dR(\xi) + \\ \int_z^h W_{n+p,\mu}(\xi,\tilde{n})\psi(\xi,\tilde{n})dR(\xi) + \\ \int_h^1 W_{n+p,\mu}(\xi,\tilde{n})M(\xi,\tilde{n})dR(\xi) + \\ \int_1^{\infty} W_{n+p,\mu}(\xi,\tilde{n})g(\xi,\hat{\Pi})dR(\xi) \end{array} \right\} \tag{3.7}$$

where, $H(\xi,\tilde{n})$, and $G(\xi,\tilde{n})$ are the same as defined by (3.6) and (3.4) respectively and $dR(\xi)$ is defined by (2.3).

Using the relation (2.1) and setting $B_n = A_n(-1)^{n+p}\, 2^{2(n+p)}(n+P)!$, we find that the equations (1.1), (1.2). (1.3), (1.4) and (1.5) transform into

$$\sum_{n=0}^{\infty} \frac{B_n\tilde{\Pi}^n}{\Gamma(v+\frac{1}{v+n+n})} L_{n+p}^{(v-\frac{1}{2})}(\frac{x^2}{4t}) = t^{-\tilde{n}}f(x,t), 0<x<y \tag{3.8}$$

$$\sum_{n=0}^{\infty} \frac{B_n t^n}{\Gamma(\mu+\frac{1}{2}+n+p)} L_{n+p}^{(v-\frac{1}{2})}(\frac{x^2}{4t}) = t^{-\tilde{n}}\phi(x,t), y < x < z \tag{3.9}$$

$$\sum_{n=0}^{\infty} \frac{B_n t^n}{\Gamma(\mu+\frac{1}{2}+n+p)} L_{n+p}^{(v-\frac{1}{2})}(\frac{x^2}{4t}) = t^{-i}\psi(x,t), z < x < h \tag{3.10}$$

$$\sum_{n=0}^{\infty} \frac{B_n t^n}{\Gamma(\mu+\frac{1}{2}+n+p)} L_{n+p}^{(v-\frac{1}{2})}(\frac{x^2}{4t}) = t^{-\tilde{n}}M(x,t), h < x < 1 \tag{3.11}$$

$$\sum_{n=0}^{\infty} \frac{B_n t^n}{\Gamma(\mu+\frac{1}{2}+n+p)} L_{n+p}^{(v-\frac{1}{2})}(\frac{x^2}{4t}) = t^{-\tilde{n}}g(x,t), 1 < x < \infty \tag{3.12}$$

Where, $t \geq \tilde{n} > 0$; and their solution is given by

$$B_n = \frac{\Gamma(\mu+\frac{1}{2})\Gamma(n+p+1)}{2^{v+\frac{1}{2}}\,\Pi\,\tilde{n}^{v+\frac{1}{2}+2n}} \left\{ \begin{array}{l} \int_0^y e^{-(\frac{\xi^2}{4\tilde{n}})} L_{n+p}^{(v-\frac{1}{2})}(\frac{\xi^2}{4\tilde{i}}) H(\xi,\tilde{n}) dR(\xi) + \\ \int_v^z e^{-\left(\frac{\xi^2}{4\tilde{n}}\right)} L_{n+p}^{(v-\frac{1}{2})}\left(\frac{\xi^2}{4\tilde{i}}\right) \phi(\xi,\tilde{n}) dR(\xi) + \\ \int_z^h e^{-\left(\frac{\xi^2}{4\tilde{n}}\right)} L_{n+p}^{(v-\frac{1}{2})}\left(\frac{\xi^2}{4\tilde{i}}\right) \psi\left(\xi,\tilde{\Pi}\right) dR(\xi) + \\ \int_h^1 e^{-\left(\frac{z^2}{4\tilde{n}}\right)} L_{n+p}^{(v-\frac{1}{2})}\left(\frac{\xi^2}{4\tilde{\Pi}}\right) M(\xi,\tilde{n}) dR(\xi) + \\ \int_1^\infty e^{-(\frac{\xi^2}{4\tilde{n}})} L_{n+p}^{(v-\frac{1}{2})}(\frac{\xi^2}{4\tilde{n}}) g(\xi,\tilde{n}) dR(\xi) \end{array} \right\} \tag{3.13}$$

where, $H(\xi,\tilde{n})$, $G(\xi,\tilde{n})$ and $dR(\xi)$ are the same as defined by (3.6), (3.4) and (2.3). The solution of the five series equations involving generalized Laguerre polynomials can be obtained independently by the above method.

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